Kink-antikink interactions in the double sine-Gordon equation and the problem of resonance frequencies

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We studied the kink-antikink collision process for the ''double sine-Gordon'' (DSG) equation in 1+1 dimensions at different values of the potential parameter $R>0$. For small values of R we discuss the problem of resonance frequencies. We give qualitative explanation of the frequency shift in comparison with the frequency of the discrete level in the potential well of isolated kink. We show that in this region of the parameter R the effective long-range interaction between kink and antikink takes place. [S1063-651X(99)03308-5]

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I. INTRODUCTION

The resonant energy exchange mechanism, which we shall consider in this paper, was originally observed in the kink-antikink collisions for the $\lambda\phi_2^4$ theory. To examine such a process, one should consider an initial configuration in the form of a kink (K) and antikink ($\bar{K}$) placed at $x=\pm x_0$ ($x_0 \gg 1$) moving toward each other with some velocities v_i. It was found that there is a critical value of the initial velocity $v_{cr} \cong 0.2598$, and at $v_i > v_{cr}$ inelastic $K\bar{K}$ scattering takes place while at $v_i < v_{cr}$, kinks and antikinks form a bound state. This bound state then decays into small oscillations [1].

Later on, when the $K\bar{K}$ collision process was studied more carefully, so-called escape windows were found [2] in the range of the initial velocities $v_i < v_{cr}$. Escape windows are merely some values of the initial velocity $v_i = v_n$ at which kinks escape to infinity after a second collision instead of forming a bound state. This phenomenon was semiquantitatively explained in Ref. [2]. The point is that the $\lambda\phi_2^4$-theory kink excitation spectrum has one zero (translation) and one nonzero (shape) mode with the frequency $\omega_1 = \sqrt{3/2}$. It was observed that the following condition is satisfied with a reasonable accuracy:

$$\omega_1 T_{12}(v_n) = \delta + 2\pi n, \qquad (1)$$

where T_{12} is the time interval between the two collisions of the kinks, n is an integer, and δ is some constant phase. During the first $K\bar{K}$ collision, a part of their kinetic energy is transferred to excitation of the kink discrete mode ω_1. Therefore, kinks cannot escape to infinity and only go away at some distance and collide again. If condition (1) is satisfied, part of the energy that is conserved in the mode ω_1 is returned back to the kink translation mode (kinetic energy)

and kinks can overcome the mutual attraction and go to infinity. This phenomenon was named ''resonant energy transfer mechanism.''

Note that ''higher-orders'' escape windows were also found. In these cases K and $\bar{K}$ escape to infinity after three or more collisions. For more detailed information about solitary wave interactions in the classical field theory, see Ref. [3].

Let us now turn our attention to the system that we shall investigate in the present paper. The double sine-Gordon (DSG) equation can be obtained from the Lagrangian of the form

$$\mathcal{L} = \frac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2 - \frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 - V(\phi) \qquad (2)$$

with the potential

$$V(\phi) = -\frac{4}{1+4|\eta|}\left(\eta\cos\phi - \cos\frac{\phi}{2}\right). \qquad (3)$$

Parameter η may be assigned any arbitrary real value ($-\infty < \eta < +\infty$). From Lagrangian (2) we get for the real scalar field $\phi(x,t)$ in (1+1) dimensions the following equation:

$$\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial x^2} + \frac{2}{1+4|\eta|}\left(2\eta\sin\phi - \sin\frac{\phi}{2}\right) = 0. \qquad (4)$$

In the present work we shall consider the range $\eta > 0$. In this case it is suitable to introduce parameter R related with η by the equality [4]

$$\eta = \frac{1}{4}\sinh^2 R.$$

Equation (4) has a static topological solution in the form of a 4π kink (antikink) [4]:

$$\phi_{K(\bar{K})}(x) = 4\pi n \pm 4\arctan\frac{\sinh x}{\cosh R}. \qquad (5)$$

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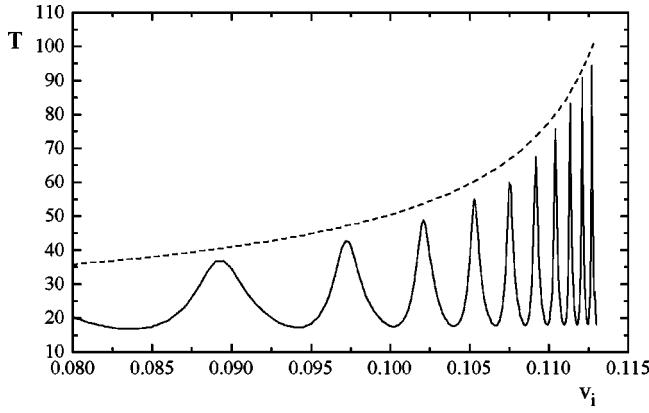[†]Electronic address: kudryavtsev@vitep5.itep.ru

FIG. 1. The time T_{12} between the first two $K\bar{K}$ collisions (dashed curve) and the time T_{23} between their second and third collisions (solid curve) as functions of the initial velocity v_i for $R = 0.5$. Dimensionless units.

The sign "+" corresponds to the case of kink, "−" corresponds to the case of antikink, and n is an integer. Equation (5) can be rewritten in the form

$$\phi_{K(\bar{K})}(x) = 4\pi n \pm [\phi_{SGK}(x+R) - \phi_{SGK}(R-x)], \qquad (5')$$

where $\phi_{SGK}(x) = 4 \arctan \exp(x)$ is the sine-Gordon (SG) equation 2π soliton. From Eq. (5a) the physical meaning of the parameter R becomes clear: DSG kink can be interpreted as a superposition of two SG solitons, separated by the distance $2R$.

The $K\bar{K}$ collision process at a variety of values of the parameter η and the initial velocity v_i was studied in detail in Ref. [4]. As for the $\lambda\phi_2^4$-theory case, there is some critical velocity v_{cr} below which kinks form a bound state decaying into small oscillations. Note that in the DSG case this critical velocity is a function of the parameter (η or R) [4].

It was found that in the DSG system the resonant energy exchange mechanism also takes place. As a consequence there is a system of escape windows at some values of η. Note that there is one important difference in kink collision processes between $\lambda\phi_2^4$ and DSG models. In the first case kinks cannot pass through each other at a valuable distance, while in the second they can travel to infinity after passing through each other. This difference is a consequence of the different structure of the potential $V(\phi)$.

In Ref. [4] different values of the parameter R were studied. At $R = 1.2$, a typical picture of escape windows was similar to the $\lambda\phi_2^4$-theory case. However, at smaller R, namely at $R = 0.5$, a new phenomenon was observed in the $K\bar{K}$ collisions — so-called quasiresonances. The essence of the phenomenon is in the following. At all velocities $v_i < v_{cr}$ we get capture and formation of the $K\bar{K}$ bound state, but the time between the second and third collisions T_{23} as a function of the initial velocity v_i has a series of well-defined maxima; see Fig. 1. Such behavior of $T_{23}(v_i)$ means that the resonant energy exchange mechanism appears in the system, but at the same time the energy that returns to the translation mode during the second collision is not enough for kinks to escape to infinity after the second collision. Besides, it turned out that the frequency of oscillations in which a part of the

kinetic energy is transferred while kinks passing through each other for the first time, i.e., ω_1 in Eq. (1), is smaller than the frequency of the localized DSG kink excitations at a given R.

It is worth mentioning that the spectrum of the small oscillations about the 4π double sine-Gordon kink was also investigated in detail in Ref. [5]. The authors of [5] used natural supersymmetry of the one-dimensional Schrödinger equation with the exactly known ground-state wave function. In [5], analytical expressions for the unnormalized eigenfunctions of the discrete and continuum modes were also presented.

II. GENERAL APPROACH

Up to now, investigating the resonant energy exchange mechanism involved the localized excitations over an isolated kink (or antikink). As it will be shown, in some cases of the $K\bar{K}$ scattering such an approximation is not valid. In these cases one should consider the spectrum of small excitations for the $K\bar{K}$ system as a whole. For this purpose, let us look for the solution of Eq. (4) for the field ϕ in the form

$$\phi(x,t) = 2\pi + \phi_K(x-x_0) + \phi_{\bar{K}}(x+x_0) + \delta\phi(x,t).$$

Such a configuration corresponds to the kink and antikink placed at $x = \pm x_0$ plus some small perturbation $\delta\phi(x,t)$, $|\delta\phi| \ll 1$. Taking into account that ϕ_K and $\phi_{\bar{K}}$ are solutions of Eq. (4), we get for $\delta\phi$ the following linearized equation:

$$\delta\phi_{tt} - \delta\phi_{xx} + \delta\phi \left.\frac{\partial^2 V}{\partial \phi^2}\right|_{\phi = 2\pi + \phi_K + \phi_{\bar{K}}} = Q(x,x_0), \qquad (6)$$

where

$$Q(x,x_0) = \left.\frac{\partial V}{\partial \phi}\right|_{\phi = \phi_K} + \left.\frac{\partial V}{\partial \phi}\right|_{\phi = \phi_{\bar{K}}} - \left.\frac{\partial V}{\partial \phi}\right|_{\phi = 2\pi + \phi_K + \phi_{\bar{K}}}. \qquad (7)$$

The explicit form of the function $Q(x,x_0)$ is rather cumbersome (see the Appendix), but nevertheless we can make several general notes. Inhomogeneity $Q(x,x_0)$ in Eq. (6) is a consequence of the fact that the configuration "kink +antikink" is not a solution of Eq. (4). The function $Q(x,x_0)$ characterizes overlapping of the kink and antikink, because $\phi_K(x-x_0)$ and $\phi_{\bar{K}}(x+x_0)$ are exact solutions of Eq. (4) when taken separately. Obviously, $Q(x,x_0)$ is an even function of x and x_0 and it falls down exponentially when x_0 increases. At fixed x_0 as a function of x, $Q(x,x_0)$ looks like two bumps with maxima at $x = \pm x_0$.

Let us now find the excitation spectrum for $\delta\phi$. For this purpose we take Eq. (6) with zero on the right-hand side and look for $\delta\phi$ in the form

$$\delta\phi(x,t) = e^{i\omega t}\chi(x).$$

Then for the function $\chi(x)$ we get the following differential equation of the Schrödinger type:

$$-\chi'' + U(x,x_0)\chi = \omega^2\chi, \qquad (8)$$

where

$$U(x,x_0) \equiv \left. \frac{\partial^2 V}{\partial \phi^2} \right|_{\phi = 2\pi + \phi_K + \phi_{\bar{K}}} . \qquad (9)$$

The explicit form of the potential $U(x,x_0)$ is rather compli-
cated (see the Appendix) and depends crucially on x_0. Note
that the shape of this potential depends on the parameter (R
or η) and $U(x,x_0) \rightarrow 1$ when $x \rightarrow \pm\infty$. Hence, $\omega < 1$ forms
the discrete excitation spectrum, and $\omega > 1$ forms the con-
tinuum one. In the limit $x_0 \gg 1$, $U(x,x_0)$ as a function of x
looks like two identical potential wells, separated by the dis-
tance $2x_0$. Each well contains one or more discrete levels,
which correspond to the localized excitations of the solitary
kink (antikink). In the collision process, DSG kinks pass
through each other, i.e., x_0 decreases to zero and then starts
to increase again. At small x_0, the distance between the
wells is small and the discrete levels are not independent.
With kinks moving toward each other from infinity, the lev-
els begin to split and then at $x_0 \lesssim 1$ the mutual potential of
the system $K\bar{K}$ is quite different from that of the solitary
kink (antikink).

Notice that in Ref. [6] the analytical expression of the
bound state frequency of the kink-kink system is presented
for the entire range of R for large distances between kinks. In
the present paper, we study numerically the behavior of the
discrete levels corresponding to the kink-antikink system ex-
citations in the range of small R and for not too large dis-
tances between kinks and antikinks.

It is worth mentioning that taking into account both wells
is also necessary in cases when in each potential well there is
a discrete level with small binding energy situated near the
continuum. In such cases one should take into account over-
lapping of the wave functions in both wells even at $x_0 \gg 1$.
This means that under some conditions, long-range interac-
tion between kinks and antikinks appears in the system.

In what follows, we will show that within such an ap-
proach the phenomenon of quasiresonances observed in the
DSG system at $R = 0.5$ in Ref. [4] may be simply explained.
We will also argue that the cause of the quasiresonances is
just the resonant energy exchange mechanism, which leads
to escape windows at some other values of R. Moreover,
there is some intermediate region of R where quasireso-
nances and escape windows appear together.

III. SMALL R

In Ref. [4], quasiresonances were observed at $R = 0.5$. We
performed similar calculations and obtained an analogous
curve $T_{23}(v_i)$; see Fig. 1. Besides that, we have investigated
the $K\bar{K}$ collision process at $R = 0.4$ and $R = 0.6$. At $R = 0.4$
(Fig. 2) we get a picture of quasiresonance peaks analogous
to the case $R = 0.5$. At $R = 0.6$ (Fig. 3) there seem to exist
escape windows in place of some peaks on the curve
$T_{23}(v_i)$. It confirms that quasiresonances and escape win-
dows are phenomena of the same nature, and with the pa-
rameter R increasing, some quasiresonance peaks transform
into the escape windows. At some intermediate values of R
both phenomena are presented, and with further increasing of
R only escape windows survive. At $R = 1.2$ in Ref. [4] a
perfect picture of escape windows and no quasiresonances
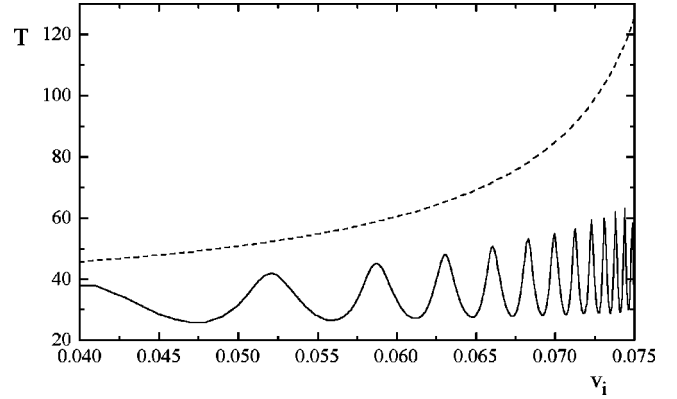was observed.

FIG. 2. The time T_{12} between the first two $K\bar{K}$ collisions
(dashed curve) and the time T_{23} between their second and third
collisions (solid curve) as functions of the initial velocity v_i for R
$= 0.4$. Dimensionless units.

To answer the question of why at given R quasireso-
nances or escape windows appear, it is required, generally
speaking, to solve Eq. (6) for $\delta\phi$ with the right-hand side. At
the same time we can suggest some truelike hypothesis. Each
bump of the source $Q(x,x_0)$ is localized on a size of order of
1 (see the Appendix). In the case of small $R \sim 0.4 - 0.6$, the
first excited level in the well is not well-localized (binding
energy is small). Therefore, the integral of overlapping of
$Q(x,x_0)$ and the wave function of the excited state is small.
It corresponds with the fact that the part of the kinetic energy
transferred to the discrete mode ω_1 is small, and hence the
loss of energy due to radiation is large. The situation changes
with increasing R. The binding energy of the first excited
level is increasing, and for $R = 1.2$ the first excited level in
the well is already well-localized. For this reason the char-
acter size of the wave function is of the same order as that of
the source. Hence the energy transfer mechanism is more
effective in this case.

From analysis of the quasiresonance peaks of the $T_{23}(v_i)$
plot for $R = 0.5$ (Fig. 1), it follows that the frequency of the
discrete mode being excited is approximately equal to $\tilde{\omega}_1$
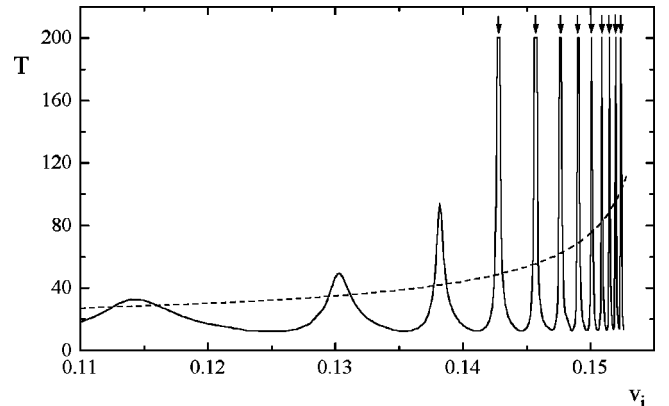$= 0.945$. At the same time, in the well corresponding to one-

FIG. 3. The time T_{12} between the first two $K\bar{K}$ collisions
(dashed curve) and the time T_{23} between their second and third
collisions (solid curve) as functions of the initial velocity v_i for R
$= 0.6$. Arrows denote probable positions of the escape windows.
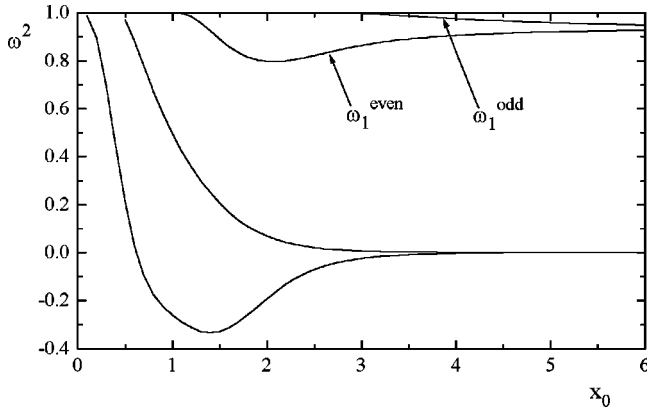Dimensionless units.

FIG. 4. The excitation frequency as a function of the initial half-distance between K and $\bar{K}$. $R=0.5$. Dimensionless units.

kink excitations there is a discrete level with frequency $\omega_1 = 0.967$. As it will be shown, this deviation is not incidental and may be easily interpreted within our approach.

At small R the system is close to the pure sine-Gordon case. Therefore, the critical velocity v_{cr} is small ($v_{cr}=0$ corresponds to the pure sine-Gordon case) and potential (9) in the Schrodinger equation (8) has one discrete level situated near the continuum (in the pure sine-Gordon case there is only zero mode). The presence of a shallow level implies that the corresponding wave function falls down slowly with the distance from the well. For this reason, while studying the $K\bar{K}$ collision process it is necessary to take into account the fact that the wells affect each other even at large distances. This leads to changes in the excitation spectrum. In Fig. 4 we show how the excitation frequency of the $K\bar{K}$ system depends on the distance between K and $\bar{K}$ (this distance is equal to $2x_0$). From the plot it is seen that even at $x_0 \gg 1$, there is some visible splitting of the higher discrete level. In the presence of the second well, this level $\omega_1 = 0.967$ splits into two sublevels: the higher with $\widetilde{\omega}_1^{odd} > 0.967$ and the lower with $\widetilde{\omega}_1^{even} < 0.967$. In the collision process the lower one is excited because the corresponding wave function is even. Moreover, the higher level may disappear during the collision. From Fig. 4 one can see that this higher level appears from the continuum $\omega > 1$ at some critical distance $2(x_0)_{cr} \gg 1$ between kink and antikink. So, if the original level in a single well lies near the continuum (as it happens at small R), then $(x_0)_{cr}$ is very large. At the same time, the level with $\widetilde{\omega}_1^{even}$ exists in a relatively wide interval of distances x_0 between kink and antikink. Namely this level is excited during the $K\bar{K}$ collision because of the resonant energy exchange mechanism. In this case the frequency ω_1 in Eq. (1) is indeed frequency $\widetilde{\omega}_1^{even}$ averaged over different x_0. It is smaller than the frequency of the discrete mode of the isolated kink, which is in correspondence with the numerical simulations.

IV. NUMERICAL CALCULATIONS

We solved the second-order partial differential equation (4) numerically on the lattice with $\Delta x=0.01$. Initial conditions were taken in the form of kink and antikink (5) situated

at $x=\pm 20$ moving towards each other with velocities $\mp v_i$, respectively. Moments of kink and antikink passing through each other were fixed via field behavior at the origin $x=0$.

To find discrete levels in the potential (9), we used the fact that the wave function falls down exponentially at large distances. We took the solution of the Schrödinger equation in the form $\chi \sim \exp(x\sqrt{1-\omega^2})$ at $x=-50$ and solved numerically stationary equation (8). As a result, we got χ at $x=50$ as a function of ω. When ω does not correspond to the discrete level, χ grows exponentially with x at positive x's, but if ω coincides with a discrete level of the potential, then χ is exponentially suppressed at large x. In real computations we observed that $\chi(x=50)$ changed its sign when ω passed a discrete level. Note that this method, which is being applied to the searching of a shallow level, does not yield a good result. In such a case one should take more distant starting and ending points. The origin of the problem is in the following: for a shallow level $\omega \to 1$, and exponents fall and grow very slowly with increasing x.

V. CONCLUSIONS

This paper presents a qualitative and semiquantitative explanation of the phenomenon of quasiresonances in collisions of kinks and antikinks of the double sine-Gordon equation at small R. It is shown that the resonant energy exchange mechanism being applied in its previous form does not give satisfactory results for frequencies.

It was shown that the resonant energy exchange between the kinks' translation mode and the discrete excitations of the $K\bar{K}$ system as a whole takes place. At small R, it is essential because of the long-range interaction in the system caused by the presence of a shallow level in the discrete spectrum of excitations of an isolated kink (antikink).

The proposed mechanism explains qualitatively the decrease of the resonance frequency ω_1 in Eq. (1) at small R in comparison with the discrete frequency of an isolated kink.

APPENDIX

In the equation

$$\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial V}{\partial \phi} = 0, \tag{A1}$$

where $V(\phi)$ has the form (3), let us substitute $\phi = 2\pi + \phi_K + \phi_{\bar{K}} + \delta\phi$. Taking into account that ϕ_K and $\phi_{\bar{K}}$ are exact solutions of Eq. (A1) and linearizing with respect to $\delta\phi$, we get

$$\frac{\partial^2 \delta\phi}{\partial t^2} - \frac{\partial^2 \delta\phi}{\partial x^2} + \left.\frac{\partial^2 V}{\partial \phi^2}\right|_{2\pi + \phi_K + \phi_{\bar{K}}} \delta\phi = Q(x, x_0), \quad (A2)$$

where $Q(x, x_0)$ is given by Eq. (7). If we substitute explicit expressions for kinks and antikinks situated at $x = \pm x_0$, respectively, then we get

$$Q(x, x_0) = \frac{8}{1 + 4\eta} \left(\frac{s_+ - s_-}{(1 + s_+^2)(1 + s_-^2)} + 4\eta \left\{ \frac{s_-(1 - s_-^2)}{(1 + s_-^2)^2} \right. \right.$$

$$\times \left[1 - \left(\frac{1 - s_+^2}{1 + s_+^2} \right)^2 \right] - \frac{s_+(1 - s_+^2)}{(1 + s_+^2)^2}$$

$$\left. \left. \times \left[1 - \left(\frac{1 - s_-^2}{1 + s_-^2} \right)^2 \right] \right\} \right), \quad (A3)$$

where

$$s_{\pm} = \frac{\sinh(x \pm x_0)}{\cosh R}$$

$[\eta$ and R are related by $\eta = (1/4)\sinh^2 R]$.

Let us use $Q(x, x_0) = 0$ in Eq. (A2) and substitute $\delta\phi(x, t) = \chi(x)\exp(i\omega t)$. Then we obtain

$$-\chi'' + U(x, x_0)\chi = \omega^2\chi,$$

where

$$U(x, x_0) \equiv \left.\frac{\partial^2 V}{\partial \phi^2}\right|_{\phi = 2\pi + \phi_K + \phi_{\bar{K}}}.$$

Insert here explicit expressions for kinks and antikinks. As a result, we have

$$U(x, x_0) = \frac{1}{1 + 4\eta} \left[\frac{1 - s_-^2}{1 + s_-^2} \frac{1 - s_+^2}{1 + s_+^2} + \frac{4s_- s_+}{(1 + s_-^2)(1 + s_+^2)} \right.$$

$$+ 8\eta \left(\frac{1 - s_-^2}{1 + s_-^2} \frac{1 - s_+^2}{1 + s_+^2} \right.$$

$$\left. \left. + \frac{4s_- s_+}{(1 + s_-^2)(1 + s_+^2)} \right)^2 - 4\eta \right]. \quad (A4)$$

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